Intermediate Math Circles Notes
Wednesday February 26, 2020
Patrick Naylor

This session will be an introduction to *mathematical games*. They seem a bit abstract, but the ideas have far-reaching applications to game theory, economics, and life! To start, we'll look at *combinatorial games*. These games are characterized by the following properties:

- Two players **take turns** (most games!);
- **No luck or chance** is involved (this rules out many card games); and
- Players have **perfect information**: in other words, any information about the game is available to both players (this also rules out many card games).

If we say that the game is *impartial*, we mean the following property:

- **Either player may make any move** (this rules out chess or checkers);

What kinds of games satisfy these properties? As it turns out, quite a few! The first one we'll look at is from the popular TV show *Survivor*.

1. **Thai 21**

   There are 21 flags. Players take turns removing 1, 2, or 3 flags. The player who takes the last flag wins.

   Solution:

   In this game, you always want to go first! This is because if you go first, no matter what your opponent does, you can arrange that the number of flags remaining is a multiple of 4. For instance, if there are 4 flags left and it's your opponents turn, *no matter what they do*, you can win! Similarly, if there are 8 flags left and it's your opponents turn, you can make sure that on their next turn, there are 4 flags left (and so on). Since there are 21 flags to start, if you go first you will always be able to win.

   If the player who takes the last flag loses, this is almost the same game! The player who takes the second last flag will win, since the other player will have no choice but to take the last one. In this case, the second player will always be able to win, since they can always make sure the number of flags is 1 plus a multiple of 4 (think about what happens when there are 5 flags left).

2. **Pick and Split**

   There are two piles of stones. One has 10 stones, and the other has 13 stones. On each players turn, they must discard one pile of stones and split the other pile into two piles. Every pile must have at least one stone. The first player unable to make a legal move loses. In other words, the game ends when there are two piles of one stone each.

   Solution:

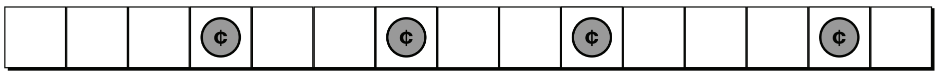
   In this game, you want to go first! This strategy is a bit tricky to come up with, but easy once you know what it is. Discard the pile of 13 stones, and divide the pile of 10 into two piles with an *odd* number of stones in each, say 7 and 3. No matter what the second player does, they must throw away one of these piles, and divide the other into two piles: one will be even, and one will odd. Once again, *no matter what they choose*, you can leave them with two piles with an odd number of stones. Eventually, they will be left with two piles of 1 stone each, and you will win.

You should convince yourself that the first player can always win as long as one of the starting piles has an even number of stones. Otherwise, the second player can always win.

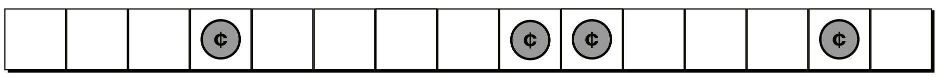
In both these games, the first player can always win. If they make their moves carefully enough, they won't lose. This isn't an accident! For any impartial combinatorial game, one of the players always has a *winning strategy.*

3. **Sliding Pennies**

The game begins with four pennies placed on a 1×15 grid, just like the picture below.



Players take turns moving one penny to the right, but may not slide pennies past each other. For instance, a legal move for Player 1 might be the following.

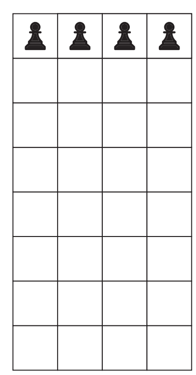


The first player unable to make a move loses. In other words, the game ends when the pennies are stacked all the way on the right side of the grid.

Solution:

In this game, you also want to go first! The winning strategy in this case is tricky to find, but like the previous problem, easy once you see it. On your first move, make the distance between the first and second, and third and fourth pennies the same (there are two ways to do this). No matter what Player 2 does, you can always make them the same again! Eventually, you will move them next to each other, and your opponent loses.

4. **Half a Chessboard**



There is a 4×8 (vertical) chessboard with four pawns in the top row. On each turn, players can move $1, 2, 3$, or all 4 pawns down one square, until they reach the bottom row. The player who cannot make a move loses.

- Do you want to go first or second?
- What if the pawns are allowed to start in any position (one in each column)?
- What if there are 5 pawns on a 5×8 chessboard? What if there are 4 pawns on an $4 \times n$ chessboard?

Solution:

Like **Sliding Pennies**, the solution to this game is about symmetry. Think about labelling the rows of the chessboard, so that the bottom is row 0, and the top is row 7. Once again, you want to go first. Move all the pawns from row 7 down to row 6. Note that no matter what your opponent does, you can always move some of the pawns so that once again, they all lie on an even numbered row. Eventually, you will move every pawn onto the bottom row (row 0), and your opponent won't be able to make a move. You win!

What are some *similarities* between the winning strategies for these games? What are some *differences*?

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky! Solutions are not provided.

5. **Erase from 13**

   A chalkboard has the numbers 1,2,3,...,13 written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

6. **Generalized Thai 21**

   There are n flags. On their turn, a player can remove 1, 2, 3,..., or k flags. The player who takes the last flag wins.

7. **Even More Thai 21**

   There are 21 flags. On their turn, a player can remove 1, 2, or 4 flags. The player who takes the last flag wins.

   There are n flags. On their turn, a player can remove 1, 3, or 5 flags. The player who takes the last flag wins.

8. **Sliding Nickels**

   Just like **Sliding Pennies**, the games begins with 5 (not 4!) nickels placed on a 1×15 grid. Players take turns sliding one nickel to the right, but may not slide nickels past each other. The first player unable to make a legal move loses.

9. **The Subtracting Game**

   Similar to **Thai 21**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *at most* as many as the previous player did. The player who takes the last flag wins.

10. **Yet Another Flag Game**

    Like **The Subtracting Game**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.

Intermediate Math Circles Problem Set
Wednesday March 4, 2020
Patrick Naylor

We looked at the following games this week, and tried to determine a winning strategy.

1. **The Calendar Game**

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 wins.

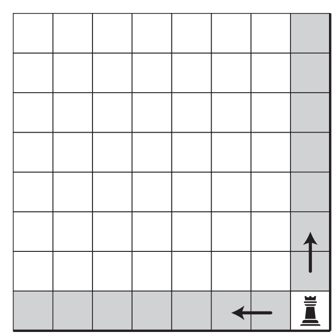
Solution:

Working backwards, notice that if you write down November 30, you will win. Indeed, no matter what your opponent chooses, you will be able to write down December 31. Similarly, if you write down October 29, no matter what your opponent chooses, you will be able to write down either November 30, or December 31. Continuing in this fashion, the same logic applies to September 28, August 27, and so on. In particular, if you go first and write down January 20, you will be able to win!

2. **A Rook on a Chessboard**

A rook is placed on the bottom right square of an 8×8 chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

- Do you want to go first or second?
- What if the game was played on an $n \times n$ chessboard?



Solution:

The solution to this game is to go second, and no matter what your opponent does, move back to the diagonal squares. Since you can't move the rook to a new diagonal square without first moving off, you will eventually win!

These games are actually the same. If we draw a labelled chessboard with months on one side and dates on the other, the squares we want to move to are exactly the diagonal ones (November 30, October 29,...). The only difference between these games is whether or not you start on the diagonal.

Sometimes, drawing a picture or a schematic makes understanding these games easier. These games are the same, but **A Rook on a Chessboard** seems much simpler.

Winning and losing positions are a useful idea that will make our study of combinatorial games easier.

A **winning position** is a position with the property that if it's your turn, you can guarantee that you can win.

A **losing position** is a position that is not a winning position. In other words, if it's your turn, you can't guarantee you'll be able to win.

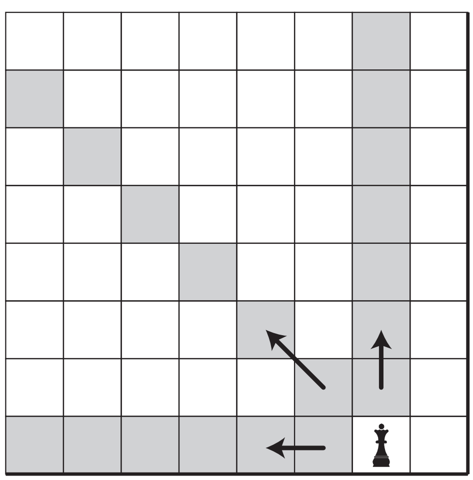
These positions actually have a lot to do with each other. Notice: there must be *at least one* move from a winning position to some losing position. In other words, there's some (good) move you can make that leaves your opponent in a losing position. Otherwise, your opponent would be able to win! Conversely, every move from a losing position must be to a winning position, or otherwise you wouldn't be able to guarantee you can win.

This is illustrated in the chessboard below: every move starting on a square labelled L ends on a square labelled W, and every square labelled W has a move to *some* square labelled L (or the top left corner).

3. **The Left Handed Queen**

   A queen is placed near the bottom right square of an 8×8 chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

   - Do you want to go first or second?
   - What if the queen starts in a different square? Can you still tell if you want to go first or second?
   - Label all the winning and losing positions on the empty chessboard below. Can you find a pattern?



	W	W	W	W	W	W	W
W	W	L	W	W	W	W	W
W	L	W	W	W	W	W	W
W	W	W	W	W	L	W	W
W	W	W	W	W	W	W	L
W	W	W	L	W	W	W	W
W	W	W	W	W	W	W	W
W	W	W	W	L	W	W	W

Solution:

Working backwards, we can label the chessboard with winning and losing positions. In particular, the starting position given is a winning position. If we add labels to the chessboard starting at zero, then the (top) losing positions have coordinates:

$$\{(1,2),(3,5),(4,7),(6,10),(8,13),(9,15),(11,18),(12,21),...\}$$

What is this sequence?

4. **Wythoff's Game**

   The game begins with a pile of 7 coins, and a pile of 11 coins. On their turn, each player may take any number of coins from either pile, or *the same number* of coins from both. The player who takes the last coin wins.

   - Do you want to go first or second?
   - Is this is the same as **The Left Handed Queen**? How can you tell?
   - What if we start with piles of 9 and 14 coins?

   Solution:

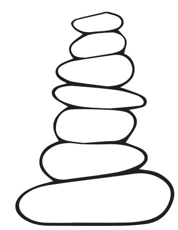
   The same way that **The Calendar Game** is the same as **A Rook on a Chessboard**, this game is the same as **The Left Handed Queen**! Label the sides of a chessboard like before, and see what happens.

Sometimes (in fact, frequently), combinatorial games don't have winning strategies that are easy to describe. A *game tree* is a useful way to understand combinatorial games. The classic game **Nim** is a good example that we'll try to understand this way.

5. **Easy Nim**

   The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

   - Do you want to go first or second? What is the winning strategy?
   - Have we seen this game before?



   Solution:

   Easy Nim isn't really **Nim**. This is just another example of a diagonal game! The winning strategy here is to go first, and make the number of stones in both piles equal. In other words, **Nim** with two piles really is easy.

6. **Actual Nim**

   The game begins with five piles of stones. There are 1, 2, 3, 4, and 5 stones in each pile. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

   - What if we only start with 1, 2, and 3 stones ($1 \oplus 2 \oplus 3$)? Is this a winning position?
   - Why is this different from **Easy Nim**?
   - Is $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ a winning position?
   - What if we started with 10 piles? With n piles?
   - What is the winning strategy? Is it easy to describe?

Solution:

The game tree for $1 \oplus 2 \oplus 3$ is drawn below. Note that from our experience with **Easy Nim**, we know that any position with two piles that are equal is a losing position. Hence, any position with a move to such a position is a winning position!

$$\begin{tikzcd} & & & 1 \oplus 2 \oplus 3 : L \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ 1 \oplus 2 : W & 1 \oplus 3 : W & 2 \oplus 3 : W & 1 \oplus 1 \oplus 3 : W \arrow[d] & 1 \oplus 2 \oplus 2 : W \arrow[d] & 1 \oplus 1 \oplus 2 : W \arrow[d] \\ & & & 1 \oplus 1 : L & 2 \oplus 2 : L & 1 \oplus 1 : L \end{tikzcd}$$

Since all moves from $1 \oplus 2 \oplus 3$ are to winning positions, it must be a losing position. Stay tuned- we'll talk about the full solution to **Nim** next week.

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

7. **Erase from 13**

   A chalkboard has the numbers $1, 2, 3, ..., 13$ written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

8. **A Knight on a Chessboard Game?**

   The game begins with a knight placed on an 8×8 chessboard. Players take turns moving the knight in the usual L-shaped moves. The player who moves it to the top left corner wins.

   - Why is this *not* a combinatorial game?

9. **A Real Knight on a Chessboard Game**

   There is a way to make the above game into a real combinatorial game. The first player *chooses* a starting position for a knight anywhere on an 8×8 chessboard. Afterwards, players take turns moving the knight to a position *that has not been visited before* (in otherwords, no repeated positions are allowed). The player who has no more available moves loses.

   - Why is this a combinatorial game?
   - This is a hard problem! Show that the second player has a winning strategy if the board has size 2×4. If you can do this, show that the second player has a winning strategy if the game is played on a 4×4 or 5×5 chessboard.
   - Can a knight placed on a chessboard visit every square exactly once?
   - If you can do this, prove that the second player has a winning strategy on a regular 8×8 chessboard.

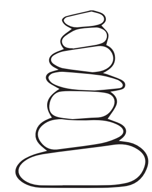
10. **Circular Nim**

    The game begins with 10 stones placed in a circle. Players take turns removing up to three *consecutive* stones. The player who takes the last stone wins.

Intermediate Math Circles Notes
Wednesday March 11, 2020
Patrick Naylor

Last week, we played the following game, but didn't talk about the solution. Here are the rules of the game:

> **Nim**
> The game begins with five piles of stones. There are $1, 2, 3, 4$, and 5 stones in each pile (to save space, we write this as $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$). On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.
>
> In fact, this game is more interesting if we allow any number of piles, with any number of stones! Now what's the winning strategy?



One thing that we noticed last week was that the winning strategy is hard to figure out! If there are only two piles, this is just one of the diagonal games from last week. In other words, $m \oplus n$ is a winning position exactly when $m \neq n$ (if this is the case, your move should be to make them equal). However, if there are three or more piles, we had to use a game tree to understand the winning and losing positions. For instance, $1 \oplus 2 \oplus 3$ is a losing position, and $1 \oplus 2 \oplus 3 \oplus 4$ is a winning position. This week, we'll discuss the full solution to **Nim**.

Solution:

One thing we'll need for the solution is the idea of a *binary number*. It's basically the same thing as a decimal number, but we're only allowed to use ones and zeros. Instead of expanding the digits of a number by powers of ten, we use powers of two (we usually include a subscript of 2 to indicate that this is a binary number). For example:

$$101101_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 32 + 8 + 4 + 1 = 45$$

To work backwards, write your decimal number as a sum of single powers of two (think about why you can always do this!). For example:

$$61 = 32 + 16 + 8 + 4 + 1 = 111101_2$$

What does this have to do with **Nim**? Let's write the number of stones in each pile in binary. For example, think about the piles $13 \oplus 4 \oplus 7$ and $11 \oplus 3 \oplus 9 \oplus 1$. In fact, these are examples of a winning and losing position. If we add up the number of ones in each column, we get the following:

$$\begin{array}{rr} 13: & 1101 \\ 4: & 100 \\ 7: & \underline{111} \\ & 1312 \end{array} \qquad \begin{array}{rr} 11: & 1011 \\ 3: & 11 \\ 9: & 1001 \\ 1: & \underline{1} \\ & 2024 \end{array}$$

Notice that in the second case ($11 \oplus 3 \oplus 9 \oplus 1$) every column had an even number of ones in it. In the first case ($13 \oplus 4 \oplus 7$), there are some columns with an odd number of ones in them. In fact, this is how we can distinguish winning and losing positions! We just have to figure out why this works.

To see that we can always go from a winning position to a losing position, think about our example above. Consider the leftmost column that has an odd number of ones in it, and pick one of the piles that has a one in it. You can change the numbers in this row so that all columns will have an even number of ones in them! For instance, we can change the 13 pile from 1101 to 0011 (or in other words, from 13 to 3).

$$\begin{array}{rr} 3: & 0011 \\ 4: & 100 \\ 7: & \underline{111} \\ & 0222 \end{array} \qquad\qquad \begin{array}{rr} 7: & 0111 \\ 3: & 11 \\ 9: & 1001 \\ 1: & \underline{1} \\ & 1024 \end{array}$$

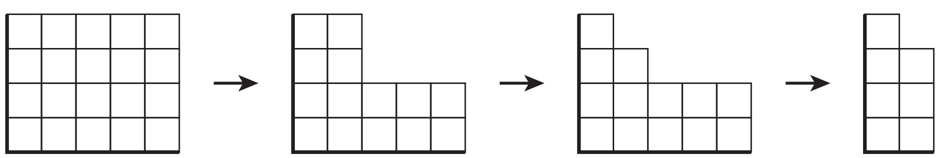
To see that any move from a losing position ends in a winning position, think about what happens if you take any number of stones away from some pile. In particular, you must change *at least one* digit in some row, but no other rows! In other words, there will be a column with an odd number of ones in it. For example, think about what happens if we change the pile of 11 to a pile of 7 (or any other number).

Once again, parity was important!

Here are some new games for this week. In each case, see if you can figure out a winning strategy. More importantly, *prove* that it is a winning strategy!

1. **Chomp**

   Like **Nim**, this is a classic combinatorial game. The game starts with a 4×5 board. Players take turns *chomping* the grid from the top right corner. In other words, they choose a point somewhere on the board, and remove all squares above and to the right of it. The player who is forced to remove (eat) the bottom left square loses. The picture below shows a possible sequence of the first few moves in a game.

   - This game is tricky, even for small boards. Draw a game tree for the 3×3 board and try to find a winning strategy. Is the full 3×3 board a winning position? Is the 4×4 board a winning position?
   - What if we use different sized boards? Are all square boards winning positions?
   - Are rectangular boards winning positions? Think about the 2×3 and 3×4 cases. Find a winning strategy if the board is of the form $2 \times n$. Can you do this for $3 \times n$ boards?
   - Prove that for *any size* rectangular board, the first player has a winning strategy.

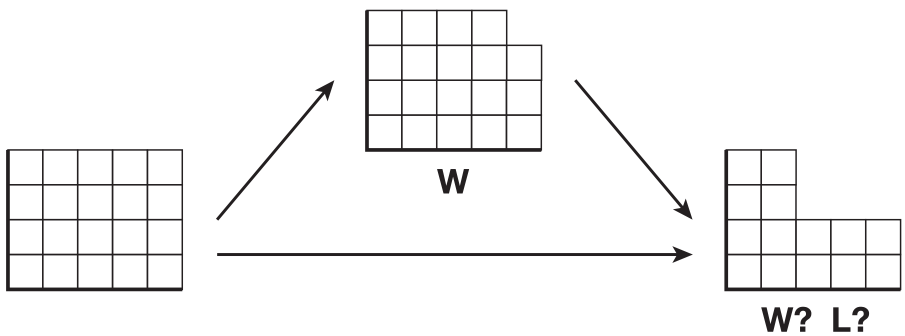
     Warning: You will have to prove this *without* describing what the strategy is explicitly. In fact, determining what the winning strategy is for arbitrary boards is an open research problem!

This game is a bit different than some of the other games we've played so far. In particular, it's much harder! One very counterintuitive thing we'll see is that the first player always has a winning strategy, but *we can't describe it.*

Solution:
Here's a short (but weird if you're not used to this kind of thing) proof that the first player has a winning strategy for any rectangular board.

Suppose that this is *not* the case. That is, the second player has a winning strategy. Then if the first player takes the top right hand corner of the board, this must be a winning position.

Now consider the result after the second player makes a move. Since they have a winning strategy, it must be a losing position. However, it isn't hard to see that whatever they do, the first player could have made this move from the beginning!



If that were the case, this position would also be a winning position. But this is impossible, since a position can't be both a winning and losing position. We conclude that the only possibility is that the second player must not have a winning strategy. Thus, the first player has a winning strategy $\square$

Notice: this proof told us nothing about what the winning strategy is. We now know that any rectangular board is a winning position, but we have no idea what move to make! However, if the board is square or has dimensions $2 \times n$, it's not too hard to explicitly give the winning strategy. You should think about it! The $3 \times n$ case is much harder. In fact, in 2002 a high school student named Steven Byrnes wrote an award winning paper titled *Poset-game periodicity* that shows that the winning positions form patterns, just like the $2 \times n$ case.

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

2. **Tic Tac Toe**

   Two friends play Tic Tac Toe. In other words, they take turns placing an $\times$ or an $\bigcirc$ on a 3×3 grid ($\times$ plays first). If a player gets three pieces in a line (horizontal, vertical, or diagonal), they win. If neither player has won by the time the grid is full, the game is declared a draw.

   - **Warning:** This isn't a combinatorial game as we've discussed them so far! Why not?
   - Show that the first player can either win or tie, *without checking cases.*
     Hint: Checking cases will eventually work, but that's not the goal here.
   - What if the game is played on an 11×11 grid? Whatever your argument is for the 3×3 case, it should work here too.

3. **Tic Tac Toe 4**

   Two friends play Tic Tac Toe, but on a 4×4 grid. Also, the first player to make 4 in a row loses!

- Show that the second player can always force a draw.

  Hint: This is different than the last question. Try to find some kind of symmetry in a 4×4 grid.

4. **Fibonacci Nim**

   This game starts with 44 coins in a pile. Player 1 can remove any number of coins, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.

   Hint: This game is named after the *Fibonacci sequence* $\{1, 2, 3, 5, 8, 13, 21, ...\}$. What is this sequence? What does it have to do with this game?

5. **Two Move Chess**

   Two friends play chess with all the usual rules, except that each players is allowed two moves per turn. Show that the first player to play *cannot lose*, that is, can always force at least a draw.

   Hint: This isn't a combinatorial game, but a strategy stealing argument will still apply.

6. **Tic Tac Toe 5**

   Two friends play Tic Tac Toe with all the same rules, but on a 5×5 grid. *Show that the second player can always force a draw.*

   Warning: Do the easier Tic Tac Toe variations first. This one is quite hard!

7. **Bishop**

   This game is played on an 8×8 chessboard. Players take turns placing a bishop anywhere on the chessboard so that it cannot be captured by the other bishops on the board. In other words, no two bishops are on the same diagonal line. The player who can no longer may a move loses. Show that the second player can always win.

8. **Knight**

   This game is played on an 8×8 chessboard. The first player *chooses* a starting position for a knight anywhere on the board. Afterwards, players take turns moving the knight to a position *that has not been visited before* (in other words, no repeated positions are allowed). The player who has no more available moves loses.

   - This is a hard problem! To start, show that the second player has a winning strategy if the board has size 2×4. If you can do this, show that the second player has a winning strategy if the game is played on a 4×4 chessboard.
   - If you can do this, prove that the second player has a winning strategy on a regular 8×8 chessboard.
   - What if the game is played on a 6×6 chessboard? Is this different from the previous questions?
   - Can a knight placed on any size a chessboard visit every square exactly once? For what sizes of chessboard is this true?