

INTERMEDIATE MATH CIRCLES PROBLEM SET
WEDNESDAY MARCH 11, 2020
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Last week, we played the following game, but didn't talk about the solution. Here are the rules of the game:

Nim

The game begins with five piles of stones. There are 1, 2, 3, 4, and 5 stones in each pile (to save space, we write this as $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$). On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

In fact, this game is more interesting if we allow any number of piles, with any number of stones! Now what's the winning strategy?



Questions:

Is $1 \oplus 2 \oplus 3$ a winning position? Is $1 \oplus 2 \oplus 3 \oplus 4$ a winning position? How can you tell?

Write down the first 10 powers of 2.

What is a *binary number*? How is this different from a decimal number?

Write the numbers 1 – 20 in *binary notation*, and write $(101101)_2$ and $(1001110)_2$ in decimal notation.

Questions:

Believe it or not, $13 \oplus 4 \oplus 7$ and $5 \oplus 7 \oplus 3 \oplus 2$ are winning positions! However, $7 \oplus 22 \oplus 17$ and $11 \oplus 3 \oplus 9 \oplus 1$ are losing positions. Write the number of stones in these piles in binary, and see if you notice any patterns. Remember what you know about **Easy Nim**!

How can we tell whether a given position in **Nim** is a winning or losing position?

Why can we always find a move from a winning position to a losing position?

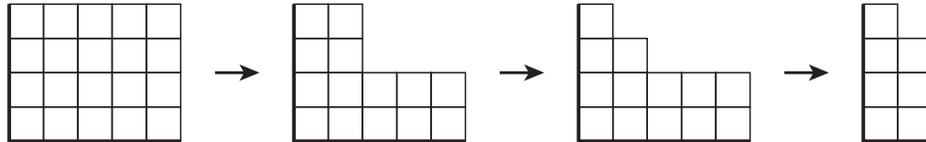
Why does every move from a losing position end in a winning position?

Is $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ a winning position? If so, what's a good move?

Here are some new games for this week. In each case, see if you can figure out a winning strategy. More importantly, *prove* that it is a winning strategy!

1. Chomp

Like **Nim**, this is a classic combinatorial game. The game starts with a 4×5 board. Players take turns *chomping* the grid from the top right corner. In other words, they choose a point somewhere on the board, and remove all squares above and to the right of it. The player who is forced to remove (eat) the bottom left square loses. The picture below shows a possible sequence of the first few moves in a game.



- This game is tricky, even for small boards. Draw a game tree for the 3×3 board and try to find a winning strategy. Is the full 3×3 board a winning position? Is the 4×4 board a winning position?
- What if we use different sized boards? Are all square boards winning positions?
- Are rectangular boards winning positions? Think about the 2×3 and 3×4 cases. Find a winning strategy if the board is of the form $2 \times n$. Can you do this for $3 \times n$ boards?
- Prove that for *any size* rectangular board, the first player has a winning strategy.

Warning: You will have to prove this *without* describing what the strategy is explicitly. In fact, determining what the winning strategy is for arbitrary boards is an open research problem!

Questions:

Why does the first player have a winning strategy in **Chomp** if the board is rectangular?

What is a *strategy stealing argument*? Why can't we necessarily describe what the winning strategy is?

Questions:

What is the winning strategy if the game is played on an $n \times n$ board?

What is the winning strategy if the game is played on a $2 \times n$ board? Does your strategy work for a $3 \times n$ board?

2. Tic Tac Toe

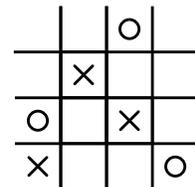
Two friends play Tic Tac Toe. In other words, they take turns placing an \times or an \circ on a 3×3 grid (\times plays first). If a player gets three pieces in a line (horizontal, vertical, or diagonal), they win. If neither player has won by the time the grid is full, the game is declared a draw.

- **Warning:** This isn't a combinatorial game as we've discussed them so far! Why not?
- Show that the first player can either win or tie, *without checking cases*.
Hint: Checking cases will eventually work, but that's not the goal here.
- What if the game is played on an 11×11 grid? Whatever your argument is for the 3×3 case, it should work here too.

3. Tic Tac Toe 4

Two friends play Tic Tac Toe, but on a 4×4 grid. Also, the first player to make 4 in a row loses!

- Show that the second player can always force a draw.
Hint: This is different than the last question. Try to find some kind of symmetry in a 4×4 grid.



More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

4. Fibonacci Nim

This game starts with 44 coins in a pile. Player 1 can remove any number of coins, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.

Hint: This game is named after the *Fibonacci sequence* $\{1, 2, 3, 5, 8, 13, 21, \dots\}$. What is this sequence? What does it have to do with this game?

5. Two Move Chess

Two friends play chess with all the usual rules, except that each player is allowed two moves per turn. Show that the first player to play *cannot lose*, that is, can always force at least a draw.

Hint: This isn't a combinatorial game, but a strategy stealing argument will still apply.

6. Tic Tac Toe 5

Two friends play Tic Tac Toe with all the same rules, but on a 5×5 grid. *Show that the second player can always force a draw.*

Warning: Do the easier Tic Tac Toe variations first. This one is quite hard!

7. Bishop

This game is played on an 8×8 chessboard. Players take turns placing a bishop anywhere on the chessboard so that it cannot be captured by the other bishops on the board. In other words, no two bishops are on the same diagonal line. The player who can no longer make a move loses. Show that the second player can always win.

8. Knight

This game is played on an 8×8 chessboard. The first player *chooses* a starting position for a knight anywhere on the board. Afterwards, players take turns moving the knight to a position *that has not been visited before* (in other words, no repeated positions are allowed). The player who has no more available moves loses.

- This is a hard problem! To start, show that the second player has a winning strategy if the board has size 2×4 . If you can do this, show that the second player has a winning strategy if the game is played on a 4×4 chessboard.
- If you can do this, prove that the second player has a winning strategy on a regular 8×8 chessboard.
- What if the game is played on a 6×6 chessboard? Is this different from the previous questions?
- Can a knight placed on any size chessboard visit every square exactly once? For what sizes of chessboard is this true?

