Intermediate Math Circles Problem Set
Wednesday March 4, 2020
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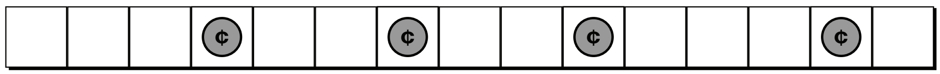
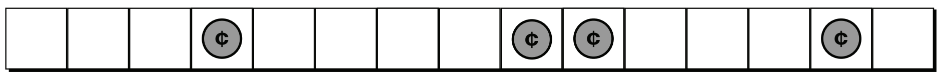
Last week, we played the following game, but didn't talk about the solution. Here are the rules of the game:

Sliding Pennies

The game begins with four pennies placed on a 1×15 grid, just like the picture below.



Players take turns moving one penny to the right, but may not slide pennies past each other. For instance, a legal move for Player 1 might be the following.



The first player unable to make a move loses. In other words, the game ends when the pennies are stacked all the way on the right side of the grid.

Questions:

What is the winning strategy for this game? How can we be sure it's a winning strategy?

What is *parity*? What does parity have to do with winning strategies?

Here are some new games for this week. In each case, see if you can figure out a winning strategy. More importantly, *prove* that it is a winning strategy!

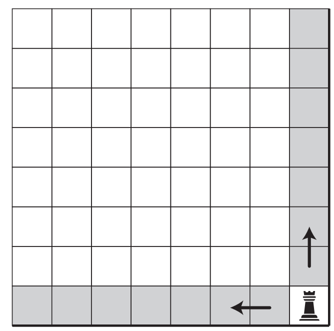
1. **The Calendar Game**

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 loses.

2. **A Rook on a Chessboard**

A rook is placed on the bottom right square of an 8×8 chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

- Do you want to go first or second?
- What if the game was played on an $n \times n$ chessboard?



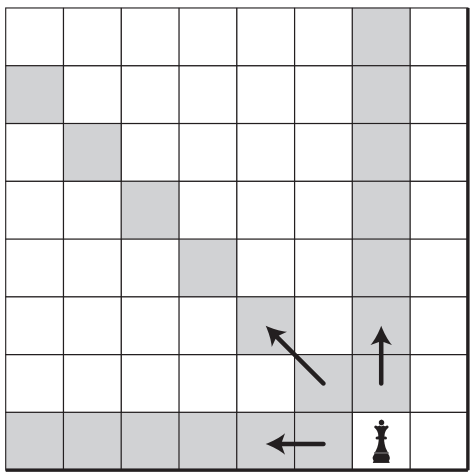
Questions:

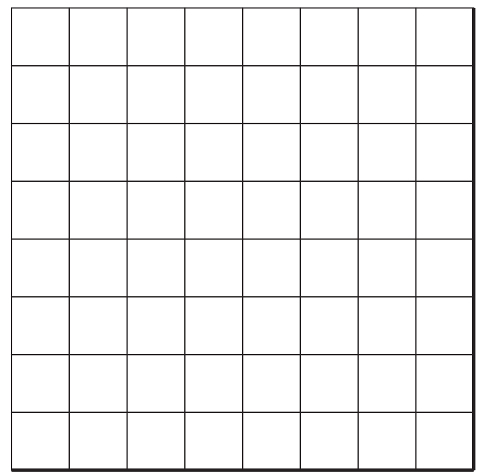
These two games seem very similar. Are they the same? How can we tell?

3. **The Left Handed Queen**

A queen is placed near the bottom right square of an 8×8 chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

- Do you want to go first or second?
- What if the queen starts in a different square? Can you still tell if you want to go first or second?
- Label all the winning and losing positions on the empty chessboard below. Can you find a pattern?





Questions:

What is a *winning position*? What is a *losing position*?

How are winning and losing positions connected?

4. **Wythoff's Game**

The game begins with a pile of 7 stones, and a pile of 11 stones. On their turn, each player may take any number of coins from either pile, or *the same number* of coins from both. The player who takes the last coin wins.

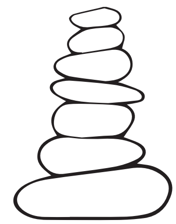
- Do you want to go first or second?
- Is this is the same as **The Left Handed Queen**? How can you tell?
- What if we start with piles of 9 and 14 stones?

Sometimes (in fact, frequently), combinatorial games don't have winning strategies that are easy to describe. A *game tree* is a useful way to understand combinatorial games. The classic game **Nim** is a good example that we'll try to understand this way.

5. **Easy Nim**

   The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player make take *any* number of stones from one pile. The player who takes the last stone wins.

   - Do you want to go first or second? What is the winning strategy?
   - Have we seen this game before?



6. **Actual Nim**

   The game begins with five piles of stones. There are 1, 2, 3, 4, and 5 stones in each pile. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

   - What if we only start with 1, 2, and 3 stones ($1 \oplus 2 \oplus 3$)? Is this a winning position?
   - Why is this different from **Easy Nim**?
   - Is $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ a winning position?
   - What if we started with 10 piles? With n piles?
   - What is the winning strategy? Is it easy to describe?

Questions:

Draw a complete game tree for $1 \oplus 2 \oplus 3$ to decide if this is a winning position.

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

7. **Erase from 13**

   A chalkboard has the numbers $1, 2, 3, ..., 13$ written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

8. **A Knight on a Chessboard Game?**

   The game begins with a knight placed on an 8×8 chessboard. Players take turns moving the knight in the usual L-shaped moves. The player who moves it to the top left corner wins.

   - Why is this *not* a combinatorial game?

9. **A Real Knight on a Chessboard Game**

   There is a way to make the above game into a real combinatorial game. The first player *chooses* a starting position for a knight anywhere on an 8×8 chessboard. Afterwards, players take turns moving the knight to a position *that has not been visited before* (in otherwords, no repeated positions are allowed). The player who has no more available moves loses.

   - Why is this a combinatorial game?
   - This is a hard problem! Show that the second player has a winning strategy if the board has size 2×4. If you can do this, show that the second player has a winning strategy if the game is played on a 4×4 or 5×5 chessboard.
   - Can a knight placed on a chessboard visit every square exactly once?
   - If you can do this, prove that the second player has a winning strategy on a regular 8×8 chessboard.

10. **Circular Nim**

    The game begins with 10 stones placed in a circle. Players take turns removing up to three *consecutive* stones. The player who takes the last stone wins.