Grade 9/10 Math Circles
March 30, 2022
Knot theory - Solutions

Exercise Solutions

Exercise 1:

The trefoil knot in Figure 1 has three crossings. How many crossings do the other knots in Figure 1 have?

Exercise 1 Solution:

From left to right, these knots have 0, 3,4,5, and 6 crossings.

Exercise 2:

Prove that any knot diagram with only two crossings is a diagram for the unknot.

Exercise 2 Solution:

If a knot diagram has only two crossings, then there are only two possibilities (up to flipping the diagram over): the crossings look the same, or one is the mirror of the other.

There are only a few ways to connect the ends of these crossings so that the resulting diagram has only one connected piece (remember, we can't introduce any other crossings). Some of the possibilities are illustrated below.

In all cases (you should check), it turns out we get something unknotted!

Exercise 3:

Find diagrams for the knot 3_1 with 4, 5, and 6 crossings. Can you find a diagram for 3_1 with 100 crossings? Why does this not contradict the fact that $c(3_1) = 3$?

Exercise 3 Solution:

The following diagram for the knots 3_1 has many extra crossings.

This does not contradict the fact that $c(3_1) = 3$; the crossing number of 3_1 is the *smallest* number of crossings possible in a diagram for 3_1. These diagrams simply have extra crossings.

Exercise 4:

The knot diagram in Figure 5 is actually unknotted! However, this is a surprisingly "hard" diagram. Can you show that this knot diagram is indeed the unknot?

Hint: Instead of trying to visualize this process, you can work with the diagram like the example below. If you have a chalkboard or a dry-erase board, this makes things even easier!

Exercise 4 Solution:

Rather than draw out a confusing series of pictures, here is a very fast animation that shows how to unknot this diagram. If you follow closely, you should be able to see what steps to take to undo this knot.

In case you're interested, these kinds of computer simulations work by giving the knot a kind of mathematical "electrical charge." The same way that particles with the same charge repel each other, this makes this knot repel itself, and eventually become unknotted.

You can also find a series of pictures unknotting this knot on page 125 of Robert Scharein's thesis here.

Exercise 5:

Show that the Perko "pair" of knots are really just the same knot.

Exercise 5 Solution:

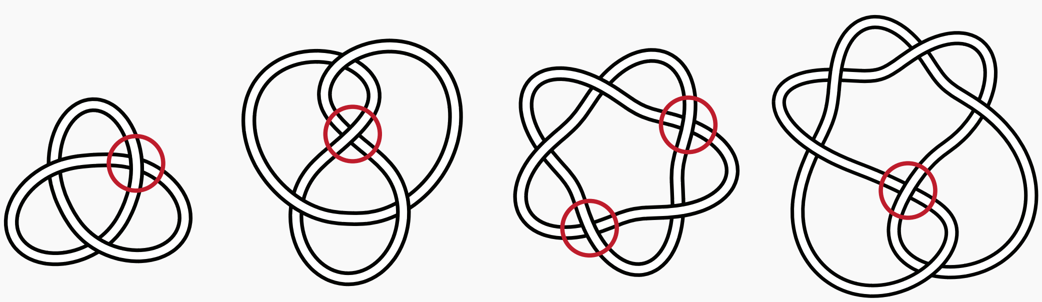
Just like Exercise 4, here is an excellent sequence of images showing that these two knots are the same.

Exercise 6:

For each of the diagrams in Figure 1 and Figure 7, find some crossing changes so that the resulting diagram is unknotted.

Exercise 6 Solution:

Here is a picture of each knot. If you change the circled crossings, the resulting diagram becomes unknotted.



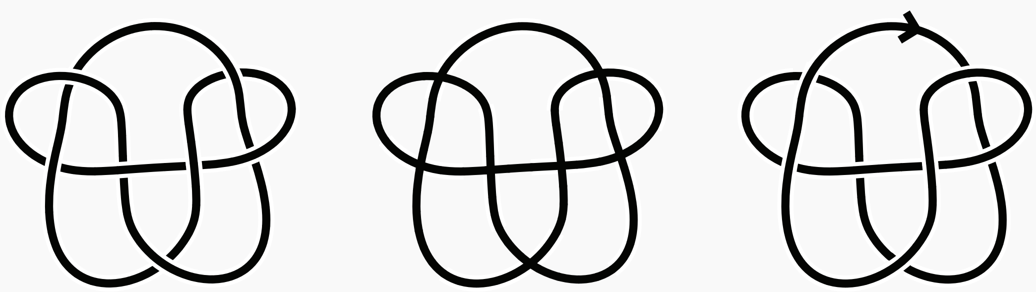
Exercise 7:

Show that given any diagram for a knot, there is always a way to change some of the crossings so that the resulting diagram is unknotted.

Hint: Pick a starting point and move around the knot in some direction. As you move around, change crossings so that you always cross *over* any part of the knot you have already visited.

Exercise 7 Solution:

This problem is a bit tricky. Rather than worry about how to change the existing crossings, consider what happens when we forget the crossings altogether. For example, starting with the knot 7_7, we get a drawing of a loop which crosses itself at some number of points (although we've forgotten whether it was an over or under crossing).



We will show that we can assign crossings to this flattened picture so that the resulting knot diagram is unknotted. Equivalently, there is a way to change some of the original crossings to get an unknotted diagram.

Pick a starting point (illustrated by the arrow) and move around the loop. As you do so, change crossings so that you are always travelling *over* any part of the loop that you have visited before. For example, for the knot 7_7 we get the diagram on the right.

The resulting knot diagram is now unknotted! To see this, imagine "picking up" the diagram from the arrow. Can you see that it must be unknotted?

Exercise 8:

Use Reidemeister moves to show that the two diagrams in Figure 11 describe the same knot.

Exercise 8 Solution:

One way to see that these knots are the same is to imagine rotating the knot 180° through an axis in the page. With this motion in mind, you can write down a sequence of Reidemeister moves taking one knot to the other.

Exercise 9:

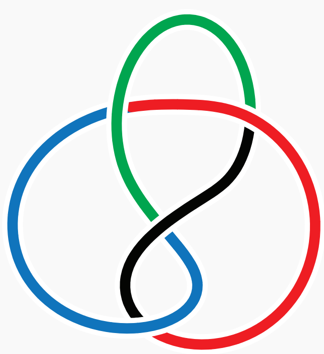
Show that the diagrams for the figure eight knot and for the unknot in Figure 1 can't be tricolored.

Exercise 9 Solution:

There is an easy reason that the unknot cannot be tricolored: by definition, any valid tricoloring must use more than one color! The simplest diagram for the unknot has no crossings, and so doesn't admit a tricoloring with more than one color.

To see that the figure eight knot cannot be tricolored, suppose that we start trying to color the diagram below. For example, we could pick one arc to be red. When we come to a crossing, we have two choices: either all of the arcs of that crossing are red, or they are all different colors.

If they're all red, then we see that the *whole* diagram must be colored red, which isn't a valid tricoloring. If we color the next arc blue, then we must also color one of the other arcs green, which only leaves one more arc.



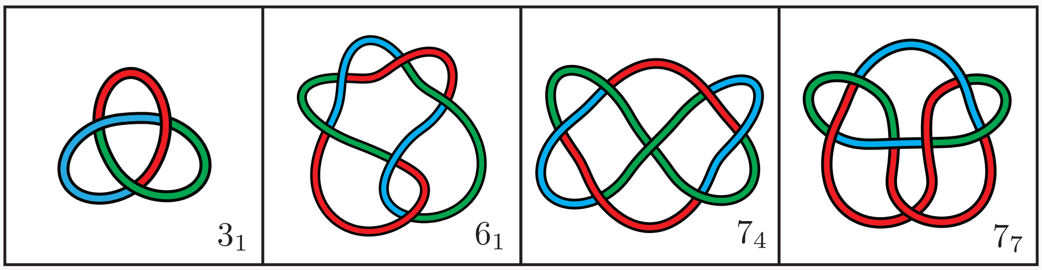
However, no matter what color we try to assign to this last arc, we don't get a valid tricoloring. Thus, the figure eight knot can't be tricolored!

Exercise 10:

What other knots in the knot table on page 11 are tricolorable?

Exercise 10 Solution:

The knots $3_1, 6_1, 7_4$, and 7_7 are tricolorable. They are each illustrated below, with valid tricolorings.



Note that these knots admit many different tricolorings, so the ones pictured here may different than the ones you found!

Problem Set Solutions

Alternating knots

1. Which diagrams in the knot table are alternating?

    Solution: They are all alternating! This is not what one expects in general– most knots don't admit an alternating projection.

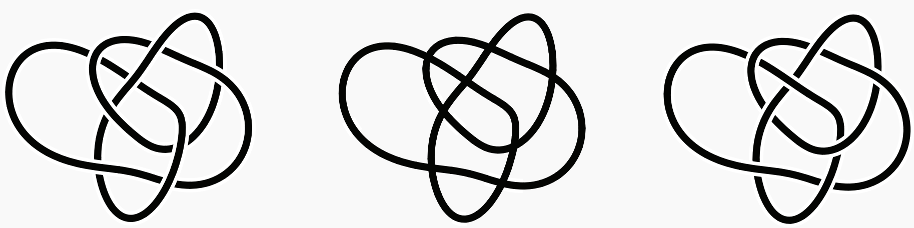
2. Find a diagram for the figure eight knot which is not alternating.

    Solution: The following diagram describes the figure eight knot (check!) but is not alternating.

There are many ways to make a diagram non-alternating, so this might look different than the one you found!

3. Show that you can change the crossings of any knot diagram to produce an alternating diagram (for a different knot).

Solution:
This problem is tricky– if you haven't looked at it already, try to do Exercise 7 from the lesson before doing this one. As in Exercise 7, we will forget about the crossings of the original knot diagram, and prove that we can assign *new* crossings so that the resulting diagram is alternating. For example, if we forget the crossings of the knot 8_{19} (not in our table) on the left of the illustration below, we obtain the drawing of a loop in the center. In fact, one can prove that 8_{19} is *not* alternating, i.e. cannot be depicted by an alternating diagram.



To get an alternating projection of some new knot, start anywhere on the loop and assign crossings that alternate between over and under crossings. If we do this for the drawing above, we get the knot on the right (which is 8_{16}).

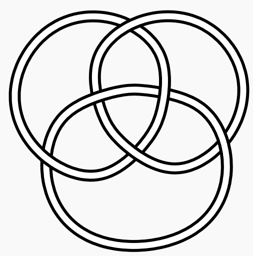
This seems easy– but there's an important question we have to answer: why does this procedure work? In other words, why is it that when we return to a crossing, it has the right sign?

To see this, note that any arc of the knot that connects a crossing C to itself intersects the knot an *even* number of times (you should check this). Thus, when we leave from and return to the crossing C, we have "alternated" an even number of times, and the crossing has the right type. In other words, we can actually implement this procedure without any issues!

Links

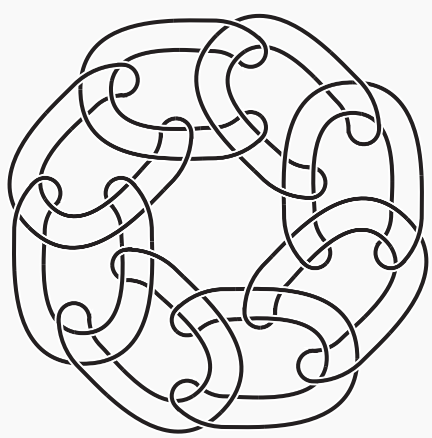
1. A (nontrivial) n-component link is called **Brunnian** if the removal of *any* component produces an $(n-1)$ component unlink. For example, the Whitehead link is a 2-component Brunnian link. Can you find a 3-component Brunnian link?

Solution: One 3-component Brunnian link is called the Borromean rings! It is illustrated below. You should check that if we remove any one of the components, the other two are unknotted.



2. For each natural number n, find an n-component Brunnian link.

Solution: The following link is Brunnian, and has 8 components. It can easily be generalized to have any number of components!



You can also find a fun video of a Brunnian link being untangled here.

3. For what values of a, b, and c is the pretzel link $P(a, b, c)$ actually a knot?

Solution:
The prezel "link" $P(a, b, c)$ is a knot as long as at least two of a, b, and c are odd! If this is the case, the three strands on both the top and bottom are connected through the twisted regions, which means that the diagram is a knot (you should check this).

Conversely, if $P(a, b, c)$ is a knot, then the three strands on the top and bottom must be joined through some of the twisted regions, which means that at least two of a, b, and c are odd.

The example $P(-3, 4, 2)$ illustrated in the problem set is a link, because we have $b = 4$ and $c = 2$, which are both even. On the other hand, $P(-3, 4, 3)$ and $P(-3, 3, 3)$ are both knots.

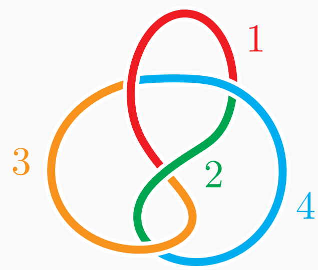
Tricolorability

1. Show that the trefoil knot and the figure eight knot are distinct knots.

Solution:
We know that the trefoil knot is tricolorable, but the figure eight knot is not! Since tricolorability is a knot invariant, these cannot be the same knots.

2(a). Show that you can label the strands of the figure-eight knot with the integers $1, 2, 3, 4$ (using at least two distinct integers) so that at each crossing, the number $x + y - 2z$ is divisible by 5 (where z is the strand on top).

Solution: A valid 4-coloring for the figure eight knot is illustrated below. There are many possible 4-colorings, so this might not be the one you came up with!



2(b). If a knot can be labelled in such a way, we say that it admits a **4-coloring**. Show that the unknot does *not* admit a Fox 4-coloring, and conclude that the figure eight knot and the unknot are distinct knots.

Solution: The unknot doesn't admit a 4-coloring for the same reason that it doesn't admit a tricoloring! The simplest diagram has no crossings, and therefore any coloring of this diagram must use only one number (which isn't a valid 4-coloring).

Since "4-colorability" is a knot invariant, the figure eight knot and the unknot must be distinct knots.

As you might guess, in general there is a notion of an n-coloring, which can also be used to distinguish knots. In this sense, tricolorings are exactly the same as 3-colorings!