Math 203, Fall 2022
Advanced Vector Calculus

Lecture #2b Wednesday, September 14, 2022

Cylindrical and Spherical Coordinates

In $\mathbb{R}^2$, we can use polar coordinates (r, θ) rather than Cartesian coordinates (x, y), where a point is specified by (r, θ), where r is the distance from the origin and θ the angle that the vector (x, y) makes with the positive x-axis.

We have

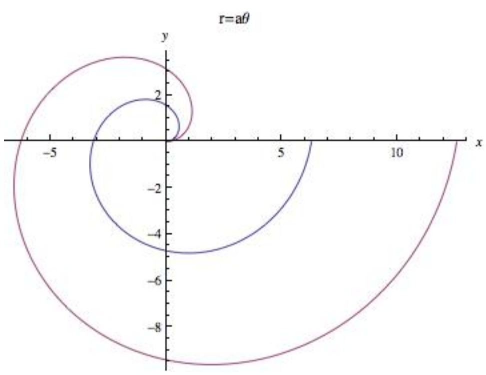
$$x = r\cos\theta, \qquad y = r\sin\theta$$

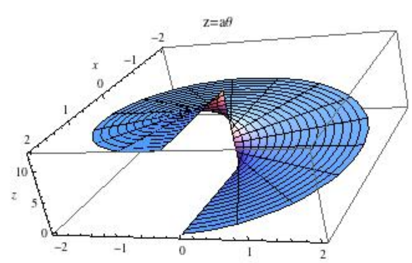
We require $r \geq 0$ and $\theta \in [0, 2\pi)$.

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \begin{cases} \arctan\frac{y}{x} & x > 0 \text{ and } y \geq 0 \\ \pi + \arctan\frac{y}{x} & x < 0 \\ 2\pi + \arctan\frac{y}{x} & x > 0 \text{ and } y < 0 \end{cases}$$

where $\arctan\frac{y}{x}$ is in $[-\pi/2, \pi/2]$. Polar coordinates can be very useful to represent circles or lines through the origin, or if there is radial symmetry.

Example 1. $r = a$ represents a circle of radius of a; $\theta = \alpha$ represents a ray starting at the origin making an angle α with positive x-axis; $r = a\theta$, $a > 0$ represents a spiral (the Archimedes spiral) starting at the origin. The smaller a is, the tighter the spiral.





Example 2. What curve does the equation $r = 2a\cos\theta$ represent?
Multiplying both sides by r, we have

$$r^2 = 2ar\cos\theta, \qquad x^2 + y^2 = 2ax.$$

$$(x - a)^2 + y^2 = a^2.$$

This is a circle of radius a centered at $(a, 0)$. When $\theta = \pi/2$, $r = 0$. Multiplying both sides of the equation by r does not introduce the origin as extra solution.

In $\mathbb{R}^3$, there are two other natural systems of coordinates: cylindrical (r, θ, z) and spherical (ρ, θ, ϕ) coordinates.

$$\textbf{Cylindrical Coordinates:} \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

A point P is specified by three coordinates (r, θ, z), where r is distance to the origin of the projection P' of P on the xy-plane, θ is the angle $\overrightarrow{OP'}$ makes with the positive x-axis, so (r, θ) is the polar coordinate for the point P' in the xy-plane, and z is just the z-coordinate of P, or height of P from the xy-plane.

What are the surfaces described by $r = a$? $\theta = \alpha$, and $z = b$? Cylindrical coordinates are useful in describing objects with radial symmetry (symmetry about a line).

Analogous to $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, the orthonormal vectors in $\mathbb{R}^3$ corresponding to cylindrical coordinates are

$$\hat{e}_r = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0) = (\cos\theta, \sin\theta, 0), \qquad \hat{e}_\theta = (-\sin\theta, \cos\theta, 0), \qquad \hat{e}_z = (0, 0, 1).$$

Example 3. $z = ar$ specifies a half cone. At height 1, the cone has radius $1/a$, so the smaller a is the more open the cone.

Example 4. The surface $z = a\theta$ is more complicated. Fix an angle α, we get a ray of height $a\alpha$ with angle α. The resulting surface is a helicoid, looking like a spiral staircase.

$$\textbf{Spherical Coordinates:} \qquad r = \rho\sin\phi, \qquad \theta = \theta, \qquad z = \rho\cos\phi.$$

$$\rho = \sqrt{r^2 + z^2}, \qquad \theta = \theta, \qquad \phi = \arccos\frac{z}{\rho}.$$

In spherical coordinates, ρ is the distance from the point P to the origin, θ is the angle with positive x-axis made by the projection of $\overrightarrow{OP}$ down to the xy-plane, and ϕ is the angle the position vector $\overrightarrow{OP}$ makes with the positive z-axis. $\rho \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. In Cartesian coordinates, we have

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\cos\theta, \qquad z = \rho\cos\phi.$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \begin{cases} \arctan\frac{y}{x} & x > 0 \text{ and } y \geq 0 \\ \pi + \arctan\frac{y}{x} & x < 0 \\ 2\pi + \arctan\frac{y}{x} & x > 0 \text{ and } y < 0 \end{cases},$$

$$\phi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

What are the surfaces described by $\rho = a$? $\theta = \alpha$, and $\phi = \beta$? Spherical coordinates are useful in describing objects with spherical symmetry (symmetry about a point).

A **great circle** on a sphere S is the intersection of S with a plane through the center. If P and Q are on S the shortest path along the sphere connecting P and Q is an arc of a great circle. To visualize this, rotate the sphere so P and Q are on the equator. On a sphere, θ corresponds to a **longitude** and ϕ corresponds to a **latitude**. Note that longitudes lie in great circles.

Analogous to $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, the orthonormal vectors in $\mathbb{R}^3$ corresponding to spherical coordinates are

$$\hat{e}_\rho = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

$$\hat{e}_\theta = (-\sin\theta, \cos\theta, 0), \qquad \hat{e}_\phi = \hat{e}_\rho \times \hat{e}_\theta = (-\cos\phi\cos\theta, -\cos\phi\sin\theta, \sin\phi).$$

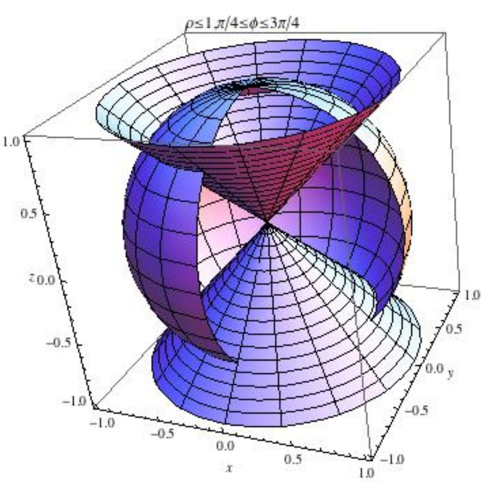
Example 5. Describe the region

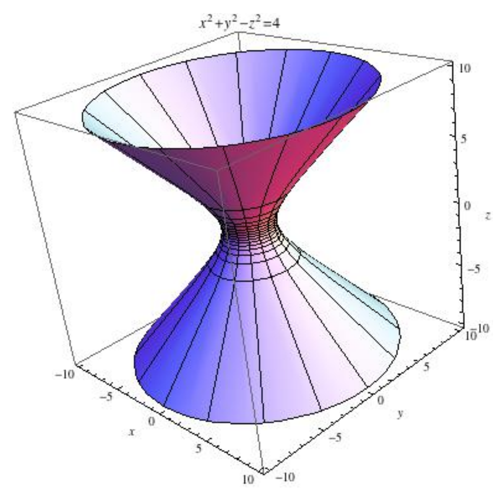
$$x^2+y^2+z^2 \le a^2 \qquad \text{and} \qquad x^2+y^2 \ge z^2$$

in spherical coordinates. The first region is inside the sphere of radius a: $\rho \le a$. The second region is given by $\sin^2\phi \ge \cos^2\phi$, or $\tan\phi \ge 1$ or $\tan\phi \le -1$, so $\pi/4 \le \phi \le 3\pi/4$. Hence, the region in common is

$$\rho \le a \qquad \pi/4 \le \phi \le 3\pi/4.$$

The region is enclosed by the ball and the cone.





Example 6. Describe the surface $x^2 + y^2 - z^2 = 4$ in spherical coordinates.

$$x^2 + y^2 - z^2 = \rho^2 - 2\rho^2 \cos^2 \phi = -\rho^2 \cos(2\phi).$$
$$\rho^2 \cos(2\phi) + 4 = 0.$$

In cylindrical coordinates, it is given by $r^2 = z^2 + 4$. It is a hyperboloid, an example of a surface of revolution, independent of θ.

n-Dimensional Euclidean Space

Definition 1.

- A **vector** $\vec{v} \in \mathbb{R}^n$ is an n-tuple of real numbers $(v_1, v_2, \cdots, v_n)$.
- The **zero vector** $\vec{0} = (0, 0, \cdots, 0)$. By convention, the zero vector is parallel and perpendicular to all vectors.
- If $\vec{v} = (v_1, v_2, \cdots, v_n)$, and $\vec{w} = (w_1, w_2, \cdots, w_n)$, then the **sum of vectors** $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w}$ is the vector $(v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$.
- The **scalar productor** of $\lambda \in \mathbb{R}$ and $\vec{v}$, $\lambda{\cdot}\vec{v}$ is the vector $(\lambda v_1, \lambda v_2, \cdots, \lambda v_n)$.

The sum and scalar product of vectors in $\mathbb{R}^n$ obey the same rules as those in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 2. The **standard basis of** $\mathbb{R}^n$ is the set of vector

$$\hat{e}_1 = (1, 0, \cdots, 0), \qquad \hat{e}_2 = (0, 1, 0, \cdots, 0), \qquad \ldots, \qquad \hat{e}_n = (0, \cdots, 0, 1).$$

If $\vec{v} = (v_1, v_2, \cdots, v_n)$, then

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots + v_n\hat{e}_n.$$

We adopt the convention that two vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if one vector is a scalar multiple of the other.

Definition 3. The **dot product** of $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is the scalar

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

The **norm** (or **length**) of $\vec{v}$ is the scalar

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

The properties of dot products of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ hold true in $\mathbb{R}^n$ as well. Of particular importance is positive definiteness, from which we can derive the following theorem.

Theorem 1 (Cauchy-Schwarz)**.** *For any two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$,*

$$| \vec{v} \cdot \vec{w} | \leq \|\vec{v}\|\|\vec{w}\|,$$

where equality holds if and only if $\vec{v}$ is parallel to $\vec{w}$.

Proof. We can assume that neither $\vec{v}$ nor $\vec{w}$ is the zero vector, otherwise, there is nothing to prove. Let $\vec{u} = x\vec{v} + \vec{w}$ for some scalar x. By positive definiteness, $\|\vec{u}\|^2 \geq 0$. Hence

$$(\vec{v} \cdot \vec{v})x^2 + 2(\vec{v} \cdot \vec{w})x + \vec{w} \cdot \vec{w} = ax^2 + bx + c \geq 0, \qquad \text{where } a, c > 0$$

The corresponding quadratic function $f(x) = ax^2 + bx + c$, being non negative has at most one real root, so the discriminant must be ≤ 0. Therefore,

$$4(\vec{v} \cdot \vec{w})^2 - 4\|\vec{v}\|^2\|\vec{w}\|^2 \leq 0, \qquad \text{or} \qquad | \vec{v} \cdot \vec{w} | \leq \|\vec{v}\|\|\vec{w}\|.$$

Equality occurs when the discriminant is zero, in which case f has a real root $\lambda \neq 0$ and hence $\vec{u} = \lambda\vec{v} + \vec{w}$ has zero length which implies that $\vec{v}$ and $\vec{w}$ are parallel. □

Definition 4. The unique **angle** θ between two non-zero vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ can be determined from

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}, \qquad 0 \leq \theta \leq \pi.$$

Example 7. Recall the triangle inequality for vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$

In particular,

$$\|\vec{v} + (\vec{w} - \vec{w})\| \leq \|\vec{v}\| + \|\vec{v} - \vec{w}\|$$
$$\|\vec{v} - \vec{w}\| = \|\vec{w} - \vec{v}\| \geq \|\vec{w}\| - \|\vec{v}\|$$

Example 8.

$$\vec{v} \cdot \vec{w} = (v_1, \cdots, v_n) \cdot (w_1, \cdots, w_n) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}, \qquad \vec{r}_j \in \mathbb{R}^n, \qquad A\vec{w} = \begin{bmatrix} \vec{r}_1 \cdot \vec{w} \\ \vec{r}_2 \cdot \vec{w} \\ \vdots \\ \vec{r}_m \cdot \vec{w} \end{bmatrix} \in \mathbb{R}^m$$

Definition 5. A function or map $F : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if for all $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and for all scalars $\lambda \in \mathbb{R}$,

$$F(\vec{v}+\vec{w}) = F(\vec{v}) + F(\vec{w}), \qquad \text{and} \qquad F(\lambda\vec{v}) = \lambda F(\vec{v}).$$

If A is an $m \times n$ matrix, and B an $n \times p$ matrix, we get functions

$$F : \mathbb{R}^n \to \mathbb{R}^m, \qquad F(\vec{v}) = A\vec{v};$$

$$G : \mathbb{R}^p \to \mathbb{R}^n, \qquad G(\vec{u}) = B\vec{u}.$$

Both f and g are linear functions on their respective domain (check!).

Example 9. $F(x) = x + 3$ is not a linear map from $\mathbb{R}$ to $\mathbb{R}$.

Example 10.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A\vec{v} = \begin{pmatrix} x+2y \\ y \end{pmatrix}.$$

A map that arises from matrix multiplication $F(\vec{v}) = A\vec{v}$ is linear. Conversely, all linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ arise from matrix multiplications.

Example 11. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^3$ is linear.

$$F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

By linearity,

$$F\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = F\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = aF\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bF\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = A\begin{bmatrix} a \\ b \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

F is represented by matrix multiplication by A.

More generally, suppose $\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n$ and $\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_m$ are standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, and F is linear, then $F(\hat{e}_j)$ is a vector in $\mathbb{R}^m$. Any vector in $\mathbb{R}^m$ is a linear combination of the standard basis vectors $\hat{f}_1, \hat{f}_2, \cdots, \hat{f}_m$, hence

$$F(\hat{e}_j) = \sum_{i=1}^{m} a_{ij}\hat{f}_i,$$

for some scalars a_{ij}.

$$F(\vec{v}) = F(\sum_{j=1}^{n} v_j \hat{e}_j) = \sum_{j=1}^{n} v_j F(\hat{e}_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} v_j \hat{f}_i = A\vec{v}.$$

The composite of two linear functions

$$F : \mathbb{R}^n \to \mathbb{R}^m, \qquad F(\vec{v}) = A\vec{v}$$

and

$$G : \mathbb{R}^p \to \mathbb{R}^n, \qquad G(\vec{u}) = B\vec{u}$$

is given by

$$F \circ G : \mathbb{R}^p \to \mathbb{R}^m, \qquad (F \circ G)(\vec{u}) = F(G(\vec{u})) = A(B\vec{u}) = (AB)\vec{u}.$$

Matrix multiplication is represented as composite of two linear function, which is also linear. Since composition of two functions is not commutative, $AB \neq BA$. Since composition of functions is associative, matrix multiplication is also associative.

See the *linear algebra* file in the "extras" section for a succinct summary of the basics of vector spaces, linear functions and determinants and why you should care.